INTERDISCIPLINARY PHD STUDENT PROJECT: *NON-ASSOCIATIVE ALGEBRAS, SUPERGRAVITY, & BEYOND*

Background

Understanding the basic structure of space, time, and matter is of paramount importance, not only from a theoretical point of view, but as technology is advancing towards smaller length scales, also from the point of view of applications. For a long time, physicists have struggled with what is perhaps the greatest challenge of all modern science, namely the unification of two of the most successful theories of the 20th century: general relativity and quantum mechanics. General relativity, famously discovered by Einstein, is a theory of gravity that has been successful in describing the universe at large length scales, while quantum mechanics becomes relevant at small length scales or high energies, where gravity can be neglected. *Quantum gravity* is the name given to the (yet unknown) theory that unifies them, and that is needed for *e.g.* answering questions about the early universe and the nature of black holes. A promising candidate for quantum gravity is M-theory, which has eleven-dimensional supergravity as its low energy limit. This is an extension of Einstein's theory with extra spacetime dimensions and gauge fields, as well as supersymmetry between bosonic and fermionic fields.

The presence of gauge fields means a redundancy in the mathematical formulation which manifests itself by a set of gauge transformations, with infinitesimal versions forming a Lie algebra. Fields that are invariant under the gauge transformations correspond to the physically independent degrees of freedom, while the gauge fields themselves transform non-trivially and are sorted out. The description of the fundamental forces in nature as gauge theories have turned out to be a very successful approach in our quest to better understand them.

It is not a far-fetched idea that concepts in differential geometry used to describe gravity in general relativity have to be generalized at the quantum level, and that the fundamental structures are algebraic rather than geometric. This is the idea behind the general framework of extended geometry, recently developed by Cederwall and Palmkvist. The gauge transformations for the extra gauge fields, of the same type as those describing non-gravitational forces in quantum field theories, are there unified with diffeomorphisms, which play the corresponding role in general relativity. Moreover, the symmetries that the Lie algebras describe are related to dualities connecting the different versions of string theory within the framework of M-theory.

In extended geometry, a certain type of gauge structure appears, called *tensor hierarchy*. They found their algebraic description in *tensor hierarchy algebras*, introduced by Palmkvist [25] as infinite-dimensional Lie superalgebra extensions of the duality Lie algebras. Originally, they were used in the context of gauge deformation of supergravity theories [22], but have proven very useful for understaning extended geometry [7–12,14–16,19].

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Recently, Cederwall and Palmkvist [13,17] discovered that underlying tensor hierarchy algebras is a non-associative structure. As it turns out, this non-associative structure also appears in generalizations of *Weyl algebras*. The Weyl algebras, most famous for the fundamental role they play in quantum mechanics [18], have moreover been generalized to the non-associative setting in different contexts by Aryapoor and Bäck [2] and by Bäck and Richter [3, 4]. Surprisingly, in the non-associative setting, more can sometimes be proven than in the associative ditto; an analogue of the famous and in still unsolved Dixmier conjecture [20] can for instance be proven true for non-associative Weyl algebras [4].

Another non-associative algebra that has just gotten much attention from physicists trying to get the grip of supergravity theories and beyond, is the algebra of *octonions* (see *e.g.* [1,5,6,21,23,24]). The octonions constitute an eight-dimensional algebra over the real numbers, and they naturally generalize the complex numbers. Moreover, in the above theories, the dimension of the algebra of octonions is precisely what is believed to decide the dimension of spacetime itself [5, 6, 23, 24].

In [2], Aryapoor and Bäck discovered a non-associative algebra that is not only responsible for the construction of some non-associative Weyl algebras as mentioned above, but also for that of the algebra of octonions. The construction of the octonions from this non-associative algebra is similar to that of tensor hierarcy algebras from an underlying non-associative structure, as discovered by Cederwall and Palmkvist. Moreover, the non-associative algebra in [2] is in fact a *superalgebra*, like the one considered by Cederwall and Palmkvist.

Project description

In this interdisciplinary project, we want to investigate what non-associative algebras can tell us about supergravity, and about quantum gravity beyond its low energy limit. In order to investigate this connection, as a first step, we want to construct new families of algebras from the above non-associative structures in a similar way to how tensor hierarchy algebras and the algebra of octonions are constructed. We then aim to study these algebras within a common framework by means of algebraic invariants, in order to see in what way they are related to one another and understand in what way their common properties may stem from their underlying non-associative algebraic structures. Concretely, this involves classifying the algebras up to isomorphism and computing their automorphism groups, Lie algebras of derivations, ideals, commutative and associative centers etc. Once this is done, we want to understand in what way these properties actually stem from the overlaying non-associative algebraic structures; to this end, we will try to lift common properties of the concrete examples of algebras to the above non-associative algebraic structures. Last, we want to investigate how these common algebraic properties actually translate into properties of supergravity, and beyond. Apart from obtaining original results, we believe that this could shed new light on known results about supergravity and its role in a consistent theory of quantum gravity.

The construction of the tensor hierarchy algebra in which the underlying non-associative algebra appears is called cartanification, and it is based on $\mathbb{Z}$-gradings. In a $\mathbb{Z}$-grading the algebra is decomposed into subspaces labelled by integer degrees, and even if there are infinitely many non-trivial ones, the structure is often captured by the part that only involves degree 0 and ± 1, called the local part.

Restricting a $\mathbb{Z}$-graded algebra to its local part yields a pre-structure known as a local algebra. In the cartanifiaction, a local Lie superalgebra $\mathscr{G}^{\llcorner}$ is then constructed from another one $\mathscr{G}^{\mathbb{L}}$, via the underlying $\mathbb{Z}$-graded superalgebra $\mathscr{G}^{\ell}$ which only has restricted associativity. Both $\mathscr{G}^{\ell}$ and $\mathscr{G}^{\llcorner}$ are then maximally extended to $\mathbb{Z}$-graded superalgebras $\mathscr{G}^{\partial}$ and $\mathscr{G}^{\ulcorner}$ in the most general way. The final steps from $\mathscr{G}^{\llcorner}$ to the Lie superalgebra which is actually the tensor hierarchy algebra consist of restricting to subalgebras and factoring out certain ideals. An important part of the project would be to investigate whether these ideals stem from ideals of $\mathscr{G}^{\ell}$. By factoring out these possible ideals, non-associative generalizations of Weyl and Clifford algebras would be obtained from $\mathscr{G}^{\ell}$. These should be compared to the non-associative generalizations of Weyl algebras constructed by Bäck and Richter.

In the well known case where $\mathscr{G}^{\mathbb{L}}$ is the local part of $A(0, n-1) = \mathfrak{sl}(n|1)$ in the consistent $\mathbb{Z}$-grading, the above non-associative generalization of a Weyl or Clifford algebra is in fact the ordinary associative Clifford algebra of signature (n, n), with a Grassmann subalgebra generated by n anticommuting elements. A differential form field can, by its derivative expansion, locally be described as an element in the tensor product of this Clifford algebra and the corresponding Weyl algebra. This description, including the exterior derivative d, can be extended to vector fields, and the Lie derivative of a vector field V with respect to another U can then be described as a derived bracket $[dU, V]$. Part of the project will be to investigate whether generalised diffeomorphisms in extended geometry can be described in a similar way, by starting with a different Lie superalgebra, where the $\mathfrak{sl}(n)$ part of $\mathfrak{sl}(n|1)$ is extended to an arbitrary (finite- or infinite-dimensional) Kac–Moody algebra $\mathfrak{g}$.

Taking the differential d into account, a natural question is whether the non-associativity in some sense can be understood as associativity up to higher homotopy, *i.e.* in terms of an A_∞ algebra, and how it is related to the L_∞ algebra that has already been used to describe the gauge structure in extended geometry. This can more generally give a better understanding of Kozsul duality and higher homotopy structures and how they can be used to describe supersymmetry and gauge structures in physics.

There are other constructions of tensor hierarchy algebras than the cartanification described above. In fact, when $\mathfrak{g}$ is infinite-dimensional, it is known that the Lie superalgebra obtained by the cartanification does not have the properties that we would expect from a tensor hierarchy algebra from a physical point of view, and we have to rely on other constructions. One of the aims of the project would be to investigate to what extent the different constructions are compatible with each other and with expectations from physics. Given a tensor hierarchy algebra, another aim would be to investigate its structure. How can the $\mathfrak{g}$-module that appears at a given $\mathbb{Z}$-degree be computed, and how does it decompose into submodules? Is there a pattern? In some cases, there are conjectures about the structure that we will try to prove.

The project has the advantage that even if some initial research questions are difficult to answer in full generality, there are many special cases that should be manageable and at the same time physically relevant and lead to possible reformulations of the questions. In this way we envisage a fruitful interplay between mathematics and physics.

Project team

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Jakob Palmkvist is associate professor and the head of subject of mathematics at Örebro University. He has a doctoral degree in fundamental physics and a docent certificate in physics, both from Chalmers University of Technology. He also has experience from postdoctoral positions at the Max Planck Institute for Gravitational Physics (Albert Einstein Institute, Germany), Université Libre de Bruxelles, Institut des Hautes Études Scientifiques (IHÉS, France) and Texas A&M University. Jakob Palmkvist's research concerns algebraic structures, in particular Lie superalgebras describing duality and spacetime symmetries in fundamental physics.

PhD candidate. We believe that there could be many suitable candidates to carry out the project at hand. The ideal candidate would either have a mathematics background with a strong interest in physics, or a physics background with a strong record in mathematics courses.

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