Determinantal Equations in Algebra and Computer Science

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Below I present two problems in commutative algebra with connections to computer science that could be part of a joint PhD in mathematics and computer science. My idea is that one, or both, of these should be combined with additional project(s) in computer science on matroids, matrix completion, Bayesian networks, or other related topics.

Both problems presented here concern systems of equations arising as determinants of variable matrices.

Matrix completion and determinantal matroids

Matroids are combinatorial objects that generalize the concept of independence that we know from linear algebra and graph theory. Matroid theory has been studied from a purely mathematical perspective since the 1930's, but there is a rich interplay with statistics and computer science, for example in network theory and coding theory. Here I describe a particular problem where matroids are linked to matrix completion.

Consider a matrix X with variable entries x_{ij}. Restricting the rank of X to be at most t, for some number t, can be done by imposing certain equations on the variables. These equations are encoded in a determinantal ideal $I_t(X)$.

Suppose we are given values for a subset Ω of the entries of X. Does Ω contain enough information to determine the remaining entries of the matrix, knowing that the rank is at most t? In other words, is Ω *uniquely completable*? It has been proved that a subset Ω is *finitely completable* if and only if $I_t(X)$ contains no equation only expressed in the variables Ω. Moreover, the finitely completable Ω's constitute the base sets of a matroid. The following are open problems.

1. What is a fast way of determining all finitely completable subsets, given the size of the matrix and the prescribed rank t? Is there for example a combinatorial description of such sets?
2. Among the finitely completable subsets, characterize the uniquely completable ones!

Partial results in this direction can be found in the references given below.

[1] D. Bernstein. Completion of tree metrics and rank 2 matrices. *Linear Algebra Appl.* 553:1–13, 2017

[2] D. Bernstein and A. Zelevinsky, Combinatorics of maximal minors. *J. Algebraic Combin.* 2:111–121, 1993

[3] A. Singer and M. Cucuringu, Uniqueness of low-rank matrix completion by rigidity theory *SIAM J. Matrix Anal. Appl.* 31:1621–1641, 2010

[4] M. Tsakiris. Results on the algebraic matroid of the determinantal variety. *Trans. Amer. Math. Soc.*377:731-751, 2024

Equations defining Bayesian Networks

A Bayesian Network is a statistical model on a collection of random variables. Conditional independence statements on the random variables are encoded by non-edges in the network, and the model can be illustrated by a directed acyclic graph. From a data science perspective one aims to invent algorithms for learning Bayesian networks from data sets. The problem I describe here is instead studying the theory of Bayesian networks using tools from algebra, and the idea is to in this way develop the theory as a whole.

Formally, the joint probability distribution is defined as the space of points corresponding to all possible outcomes on the random variables. Algebraically, these points are solutions to a system of polynomial equations. More precisely, each conditional independence statement gives rise to a matrix where the entries are sums of variables, and from which one extracts the equations by computing all 2-minors. The following are open problems in this area.

1. Characterize the Bayesian networks for which the polynomial equations are *binomial* equations, that is have only two terms.
2. Characterize the Bayesian networks for which the polynomial equations are quadratic. In this case, is there a Gröbner basis for the equations?

Algebras defined by binomial equations, or equations possessing a quadratic Gröbner basis, are families of well-studied algebras, meaning that there are more algebraic tools available for studying Bayesian networks of this type.

Partial results on these problems can be found in the references listed.

[1] E. Duarte and C. Görgen. Equations defining probability tree models. *J. Symb. Comput.*99:127–146, 2020

[2] L. D. Garcia, M. Stillman, and B. Sturmfels. Algebraic geometry of Bayesian Networks, *J. Symbolic Comput.*39:331–355, 2005

[3] C. Görgen, A. Maraj, and L. Nicklasson. Staged tree models with toric structure *J. Symb. Comput.*113:242–268, 2022.

[4] L. Nicklasson. Toric and Non-toric Bayesian Networks. *SIAM Journal on Applied Algebra and Geometry*7(3):549–566, 2023.