The Crouzeix conjecture

Bartosz Malman

February 2, 2024

If A is a given square matrix of dimension $n \in \mathbb{N}$, then *the spectrum* $\sigma(A)$ is defined as the set of eigenvalues of A, that is, the set of numbers $\lambda \in \mathbb{C}$ which appear in solutions to equations of the form $Ax = \lambda x$, where $x \in \mathbb{C}^n$ is a non-zero vector. The spectrum of A contains loads of information about the matrix A, but it is invariant under changes of base: if T is an invertible matrix of same dimension, then an equivalent matrix $\tilde{A} = T^{-1}AT$ has the same spectrum as A. The *numerical range* $W(A)$ is the larger set

$$W(A) := \{(Ax, x) : \|x\| = 1\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm of the vector x, and $(\cdot, \cdot)$ denotes the usual Hermitian scalar product on $\mathbb{C}^n$. The numerical range is not invariant under changes of base, and so contains more information than the spectrum about the exact structure of the matrix A. Indeed, the concept of numerical range finds applications in engineering and numerical analysis, for instance with purposes of spectral estimations.

The, now famous, numerical analyst Michel Crouzeix found in [1] a striking property of the numerical range which was (supposedly) surprising even for the applied practitioners of his field. Namely, if $p(z)$ is a complex polynomial, then the operator norm $\|p(A)\|$ satisfies

$$\|p(A)\| \leq C \sup_{z \in W(A)} |p(z)|,$$

where the constant $C > 0$ is completely independent of both the dimension n and A. He conjectured that the optimal value equals precisely 2. This assertion has been known as the *Crouzeix conjecture*, and remains unsolved. In 2007, Crouzeix proved that the constant $C = 11.08$ works, and in his phenomenal work [2] with Palencia in 2017, this was improved to $C = 1 + \sqrt{2}$.

The Crouzeix conjecture is one of the better recognized conjectures in the field of matrix analysis. Recently, Greenbaum and Overton in [3] provided numerical evidence for the conjecture. This work reassured mathematicians in validity of Crouzeix's claim, and spawned additional questions.

This research project involves further work on the conjecture and related subjects. There exists possibility for some numerical work, of the type appearing in [3]. For instance, there is an interest from pure mathematicians in a numerical verification of several related claims and conjectures.

References

[1] Michel Crouzeix. Bounds for analytical functions of matrices. *Integral Equations and Operator Theory*, 48:461–477, 2004.

[2] Michel Crouzeix and César Palencia. The numerical range is a (1+2)-spectral set. *SIAM Journal on Matrix Analysis and Applications*, 38(2):649–655, 2017.

[3] Anne Greenbaum and Michael L Overton. Numerical investigation of crouzeix's conjecture. *Linear Algebra and its Applications*, 542:225–245, 2018.